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ON MEROMORPHIC UNIVALENT FUNCTIONS OMITTING A DISC

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ABSTRACT

The class Σ_b is defined to consist of meromorphic univalent functions H omitting a disc with the radius $b: H(z) = z + \sum_0^\infty A_n z^{-n}$, $|z| > 1$, $|H(b)| > b \in (0, 1)$. By aid of FitzGerald inequalities the inverse coefficients of odd Σ_b-functions are maximized. The result extends the corresponding estimation, due to Netanyahu and Schober, from $b = 0$ to the whole interval $(0, 1)$.

1. Introduction

In what follows we shall consider certain classes of holomorphic and univalent functions defined either in the unit disc $D = \{z : |z| < 1\}$ or in the domain $\tilde{D} = \{z : |z| > 1\}$. First we shall define the classes in question.

S is the class of functions F holomorphic-univalent in D having the normalized expansion

$$F(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

$S(b)$ is the class of functions f which are holomorphic-univalent in D and bounded by 1, $|f(z)| < 1$, with the expansion

$$f(z) = b\left(z + \sum_{n=2}^{\infty} a_n z^n\right).$$

Next, introduce the class Σ_b of functions H, holomorphic-univalent in $\tilde{D}$ except at the point of infinity where they have a pole. Let the functions H be

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restricted so that

$$|H(z)| > b, \qquad 0 < b < 1,$$

and let them have the expansion

$$H(z) = z + \sum_{n=0}^{\infty} A_n z^{-n}. \tag{1}$$

As a limit case $b \to 0$ we obtain from Σ_b the class Σ of holomorphic-univalent functions H in $\tilde{D}$ with the expansion (1) and the restriction $H(z) \neq 0$.

Clearly, there is a one-to-one correspondence between the bounded classes Σ_b and $S(b)$:

$$f(z)H(1/z) = b, \qquad |z| < 1.$$

In [2] Goluzin rederives Grunsky's inequality for S-functions by applying a special summation technique in connection with Löwner's equation of the first kind. FitzGerald [1] has extended Goluzin's method by "exponentiating" certain functionals in S. This allowed him to obtain sharp results for some coefficient problems for which the Grunsky inequality is not strong enough. By his method FitzGerald has estimated among others the coefficients of the inverse functions of S.

In [3] Launonen applied FitzGerald's ideas for the exponentiation of functionals in the class $S(b)$. He applied the method first for the coefficients of functions which are inverse to odd $S(b)$-functions. By aid of the inequalities found he estimated also the coefficients of functions which are inverse to any $S(b)$-functions. Launonen replaced the original summation formalism by a more convenient integral formalism. This considerably shortened the calculations, allowing also determination of the uniqueness of the extremal functions.

In this paper, Launonen's inequality will be applied to finding sharp inequalities for the coefficients of functions inverse to odd Σ_b-functions. These estimates, unfortunately, do not imply sharp inequalities to inverse coefficients of any Σ_b-function. Launonen's inequality yields results only for a very limited amount of them.

Schober considers in [5] the class $\Sigma' \subset \Sigma$ with the additional normalization $A_0 = 0$ and maximizes odd coefficients of functions inverse to Σ'-functions. In [4] Netanyahu succeeded in estimating all the coefficients inverse to Σ-functions through an ingenious use of variational method.

Let us finally mention that Singh considers in [6] some coefficient problems in the class $\Sigma_b' \subset \Sigma_b$ in which $A_0 = 0$.

2. The FitzGerald-type condition for Σ_b-functions

In [3] Launonen proved that for every function $f(z)$ of $S(b)$ the inequality

$$\left|\int_{\gamma_1}\int_{\gamma_1} V_1(\zeta)V_1(\eta)\frac{f(\zeta)f(\eta)(\zeta-\eta)}{\zeta\eta b(f(\zeta)-f(\eta))}d\zeta d\eta\right| \tag{2}$$
$$\leqq \int_{\gamma_1}\int_{\gamma_1} V_1(\zeta)\overline{V_1(\eta)}\frac{1-f(\zeta)\overline{f(\eta)}}{1-\zeta\bar{\eta}}d\zeta d\bar{\eta}$$

holds, where γ_1 is a closed analytic curve and $V_1(z)$ is a continuous weight function on γ_1.

The connection between the classes $S(b)$ and Σ_b allows writing the inequality (2) in the form

$$\left|\int_{\gamma_1}\int_{\gamma_1} V_1(\zeta)V_1(\eta)\frac{1/\zeta-1/\eta}{H(1/\zeta)-H(1/\eta)}d\zeta d\eta\right|$$
$$\leqq \int_{\gamma_1}\int_{\gamma_1} V_1(\zeta)\overline{V_1(\eta)}\frac{1-b^2/H(1/\zeta)\overline{H(1/\eta)}}{1-\zeta\bar{\eta}}d\zeta d\bar{\eta}.$$

Applying the inversion $\zeta = 1/z$, $\eta = 1/\omega$ and setting $\gamma_1(\zeta^{-1}) = \tilde{\gamma}(z)$ and $\zeta^{-2}V_1(\zeta^{-1}) = V(z)$ we get for any function $H(z)\in\Sigma_b$

$$\left|\int_{\gamma}\int_{\gamma} V(z)V(\omega)\frac{z-\omega}{H(z)-H(\omega)}dz d\omega\right| \tag{3}$$
$$\leqq \int_{\tilde{\gamma}}\int_{\tilde{\gamma}} V(z)\overline{V(\omega)}\frac{1-b^2/H(z)\overline{H(\omega)}}{1-1/z\bar{\omega}}dz d\bar{\omega}.$$

Here $\tilde{\gamma}$ is any closed analytic curve and V is a continuous weight function.

Let now $z = I(w)$ denote the function inverse to the $H(z)$-function and let us denote $V(I(w))\cdot I'(w) = \mu(w)$ and $\tilde{\gamma}(I(w)) = \gamma(w)$. From (3) we have

$$\left|\int_{\gamma}\int_{\gamma} \mu(w)\mu(s)\frac{I(w)-I(s)}{w-s}dw ds\right| \leqq \int_{\gamma}\int_{\gamma} \mu(w)\overline{\mu(s)}\frac{1-b^2/w\bar{s}}{1-1/I(w)\overline{I(s)}}dw d\bar{s}. \tag{4}$$

We shall prove that (4) is the sharp inequality in the class of functions inverse to Σ_b-functions and that the extremal function is the function $z = I_0(w)$ satisfying the equation

$$I_0(w) + I_0(w)^{-1} = b^2 w^{-1} + w. \tag{5}$$

For this purpose, let us assume that the function $w = \bar{F}(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is the right radial-slit mapping satisfying the equation

$$\frac{\tilde{F}(z)}{(1+\tilde{b}\tilde{F}(z))^2}=\frac{z}{(1+z)^2}, \qquad |z|<1, \quad 0<\tilde{b}<1, \tag{6}$$

and the function

$$z=\tilde{G}(w)=w+\sum_{n=2}^{\infty} C_n w^n$$

is the inverse to $\tilde{F}(z)$. From [8] (p. 29) we have

$$C_n=\gamma_{n0}+\gamma_{n1}\tilde{b}+\gamma_{n2}\tilde{b}^2+\cdots+\gamma_{n(n-1)}\tilde{b}^{n-1}, \qquad n=1,2,\ldots, \tag{7}$$

$$\gamma_{nv}=(-1)^v\frac{2}{v!}\frac{n-v}{n-v+1}\frac{(2n-v-1)!}{[(n-v)!]^2}, \qquad v=0,1,\ldots,n-1. \tag{8}$$

In view of (6) the function

$$z=G_0(w)=w+\sum_{n=1}^{\infty} B_{2n+1}^0 w^{2n+1} \tag{9}$$

which is inverse to

$$w=F_0(z)=\sqrt{\tilde{F}(z^2)}, \tag{10}$$

satisfies the equation

$$G_0(w)(1+b^2w^2)=w(1+G_0^2(w)), \qquad b=\tilde{b}^{1/2}, \tag{11}$$

and from (10) we have

$$G_0(w)^2=\tilde{G}(w^2). \tag{12}$$

From (10) and (6) it follows moreover that the function

$$H_0(z)=1/F_0(1/z)$$

belonging to the class Σ_b and mapping $|z|>1$ onto $|w|>b$ minus two symmetric slits along the real axis satisfies the equation

$$H_0(z)(1+z^2)=z(H_0(z)^2+b^2).$$

The function inverse to $H_0(z)$ is

$$I_0(w)=w+\sum_{n=0}^{\infty} D_{2n+1}^0 w^{-(2n+1)}$$

given by (5).

Next, derive the coefficients D^0_{2n+1} of the function $I_0(w)$. Representing both sides of equation (11) as power series in a neighborhood of the point $w = 0$, comparing the terms and by using (12), we obtain

$$C_n = B^0_{2n+1} + b^2 B^0_{2n-1}, \qquad n = 1,2,\ldots. \tag{13}$$

From (13), (7) and from the fact that $\tilde{b} = b^2$ it follows that

$$B^0_{2n+1} = \sum_{k=0}^{n} \beta_{nk} b^{2k}, \qquad n = 1,2,\ldots, \tag{14}$$

where

$$\beta_{n0} = \gamma_{n0},$$

$$\beta_{nk} = \gamma_{nk} - \beta_{(n-1)(k-1)}, \qquad k = 1,2,\ldots,n-1,$$

$$\beta_{nn} = -\beta_{(n-1)(n-1)}.$$

Applying induction and using (8) it follows that

$$\beta_{nk} = (-1)^k \frac{2}{k!} \frac{2n-k}{2(n-k+1)} \frac{(2n-k-1)!}{[(n-k!)!]^2}, \quad n = 1,2,\ldots, \quad k = 1,2,\ldots,n. \tag{15}$$

Note moreover that from (5) we obtain

$$I_0(w) = \frac{b^2 w^{-1} + w}{2} + \frac{\sqrt{1 + (2b^2 - 4)w^{-2} + b^4 w^{-4}}}{2w^{-1}} \tag{16}$$

Analogously from (11) we have

$$G_0(w) = \frac{b^2 w + w^{-1}}{2} - \frac{\sqrt{1 + (2b^2 - 4)w^2 + b^4 w^4}}{2w}$$

whence in view of (9)

$$-\frac{\sqrt{1 + (2b^2 - 4)w^2 + b^4 w^4}}{2w} = -\frac{1}{2w} - \frac{b^2 - 2}{2} w + \sum_{n=1}^{\infty} B^0_{2n+1} w^{2n+1}. \tag{17}$$

By aid of (17) and (16) we obtain correspondingly

$$I_0(w) = w + (b^2 - 1)w^{-1} - \sum_{n=1}^{\infty} B^0_{2n+1} w^{-(2n+1)}.$$

Thus

$$\begin{cases} D^0_1 = -(1 - b^2), \\ D^0_{2n+1} = -B^0_{2n+1}, \quad n = 1,2,\ldots, \end{cases} \tag{18}$$

where B^0_{2n+1} is given by (14) and (15). Proceeding analogously we can prove that

$$\frac{1}{I_0(w)} = w^{-1} + \sum_{n=1}^{\infty} B^0_{2n+1} w^{-(2n+1)}. \tag{19}$$

Let us put now in (4)

$$\mu(w) = w^n$$

where n is any fixed positive integer. From (5) and the fact that all the coefficients of $I_0(w)$ are real it follows that

$$\int_\gamma \int_\gamma \mu(w)\mu(s)\frac{I_0(w)-I_0(s)}{w-s}\,dwds$$

$$= \int_\gamma \int_{-\gamma} \mu(w)\mu(\bar\sigma)\frac{1-b^2/w\bar\sigma}{1-1/I_0(w)I_0(\bar\sigma)}\,dwd\bar\sigma$$

$$= -\int_\gamma \int_\gamma \mu(w)\overline{\mu(\sigma)}\frac{1-b^2/w\bar\sigma}{1-1/I_0(w)\overline{I_0(\sigma)}}\,dwd\bar\sigma.$$

Thus we have shown

THEOREM 1. *For every function $z = I(w)$ inverse to the Σ_b-function, $0 < b < 1$, the inequality*

$$\left|\int_\gamma \int_\gamma \mu(w)\mu(s)\frac{I(w)-I(s)}{w-s}\,dwds\right| \leqq \int_\gamma \int_\gamma \mu(w)\overline{\mu(s)}\frac{1-b^2/w\bar s}{1-1/I(w)\overline{I(s)}}\,dwd\bar s \tag{20}$$

holds, where γ is any closed analytic curve and μ is an analytic weight function. An extremal function is the function given by (5).

By passing to the limit with $b \to 0$ in the theorem we obtain an analogous theorem in the class Σ.

3. The sharp bounds for the coefficients of functions inverse to odd Σ_b-functions

We shall now apply Theorem 1 to estimate the coefficients of any odd function

$$I(w) = w + \sum_{n=0}^{\infty} D_{2n+1} w^{-(2n+1)}. \tag{21}$$

We prove that

$$|D_{2n+1}| \leqq |D^0_{2n+1}|.$$

For this purpose, let us put in (20) the weight function $\mu(w) = w^{k-1}$, k any fixed natural number. Then the left-hand side of the considered inequality equals $4\pi^2 |D_{2k-1}|$. In order to derive the right-hand side let us put $t = w\bar{s}$ and notice that the effective term yelding the value of the right-hand side of (20) is the coefficient of the power t^{-n} of the development of the function

$$
\begin{aligned}
&[1 - b^2(w\bar{s})^{-1}]\frac{1}{1 - 1/I(w)\overline{I(s)}} \\
(22)\qquad &= (1 - b^2 t^{-1})[1 + (I(w)\overline{I(s)})^{-1} + (I(w)(\overline{I(s)}))^{-2} + \cdots]
\end{aligned}
$$

(Observe that all the other terms of this development yield the zero contribution in the integration of (20).) The coefficient of t^{-n} can be determined from Table 1, where

$$
\frac{1}{I(w)} = w^{-1} + \sum_{n=1}^{\infty} \alpha_{2n+1} w^{-(2n+1)}. \tag{23}
$$

It follows from (21) and (23) that

$$
\alpha_{2n+1} = -(D_1\alpha_{2n-1} + D_3\alpha_{2n-3} + \cdots + D_{2n-1}), \qquad n = 1, 2, \ldots.
$$

From Table 1 it follows moreover that positive and negative terms in the coefficient of t^{-k} of the function (22) are matched to yield a polynomial of α_k and $\bar{\alpha}_k$ with only positive terms. Thus, for example in the coefficient of t^{-7}, there is the term

$$
|(2\alpha_5 + \alpha_3^2) + 2\alpha_3^2 + \alpha_5|^2 - b^2|2\alpha_5 + \alpha_3^2|^2
$$

TABLE 1

n	-1	-2	-3	-4	-5	-6	-7
$(I(w)\overline{I(s)})^{-1} =$	t^{-1}		$+\lvert\alpha_3\rvert^2 t^{-3}$		$+\lvert\alpha_5\rvert^2 t^{-5}$		$+\lvert\alpha_7\rvert^2 t^{-7}$
$(I(w)\overline{I(s)})^{-2} =$		t^{-2}		$+\lvert 2\alpha_3\rvert^2 t^{-4}$		$+\lvert 2\alpha_5 + \alpha_3^2\rvert^2 t^{-6}$	
$(I(w)\overline{I(s)})^{-3} =$			t^{-3}		$+\lvert 3\alpha_3\rvert^2 t^{-5}$		$+\lvert 3\alpha_5 + 3\alpha_3^2\rvert^2 t^{-7}$
$(I(w)\overline{I(s)})^{-4} =$				t^{-4}		$+\lvert 4\alpha_3\rvert^2 t^{-6}$	
$(I(w)\overline{I(s)})^{-5} =$					t^{-5}		$+\lvert 5\alpha_3\rvert^2 t^{-7}$
$(I(w)\overline{I(s)})^{-6} =$						t^{-6}	
$(I(w)\overline{I(s)})^{-7} =$							t^{-7}

and the last two terms in the coefficient of t^{-k} equal to

$$[(k-2)^2-(k-3)^2b^2]|\alpha_3|^2+1-b^2.$$

Thus the inequality (20) can be represented in the form

$$4\pi^2|D_{2k-1}|\leqq\int_\gamma\int_\gamma t^{k-1}T(t)dwd\bar{s}, \tag{24}$$

where $t=w\bar{s}$, and

$$T(t)=1+(1-b^2)t^{-1}+(1-b^2)t^{-2}+[|\alpha_3|^2+1-b^2]t^{-3}+[(4-b^2)|\alpha_3|^2+1-b^2]t^{-4}$$
$$+\cdots+\{\cdots+[(k-2)^2-(k-3)^2b^2]|\alpha_3|^2+1-b^2\}t^{-k}+\cdots.$$

The dots in the { }-expression indicate a sum of the above type determined by the coefficients $\alpha_{2k-3},\alpha_{2k-5},\dots,\alpha_3$ with only nonnegative terms.

The condition (24) for $k=1$ yields

$$|D_1|\leqq 1-b^2=-D_1^0. \tag{25}$$

This inequality is found also (using the connection between $S(b)$ and Σ_b) as a consequence of the Power inequality [7]. This implies also that the only extremal function is the function $I_0(w)$ given by (5) and its rotations.

Taking next $k=2$ in (24), we have

$$|D_3|\leqq 1-b^2=-D_3^0,$$

but in this case the question of the uniqueness of the extremal function remains open.

For $k=3$ the inequality (24) takes the form

$$|D_5|\leqq|\alpha_3|^2+1-b^2,$$

and because

$$|\alpha_3|=|D_1|\leqq|D_1^0|=1-b^2=\alpha_3^0, \tag{26}$$

where α_k^0 denotes the k-th coefficient of the development of $I_0(w)^{-1}$, then

$$|D_5|\leqq(\alpha_3^0)^2+1-b^2=-\alpha_3^0D_1^0-D_3^0=\alpha_5^0. \tag{27}$$

Thus in view of (27) and (19)

$$|D_5|\leqq\alpha_5^0=-D_5^0,$$

where again (considering (26) and (27)) the equality holds only for I_0 and its rotations.

Let us assume now that

$$|D_{2k-3}| \leqq -D^0_{2k-3},$$

$$|\alpha_{2k-3}| \leqq \alpha^0_{2k-3}, \qquad k = 2,3,\ldots,n,$$

and estimate the right-hand side of (24) upwards by using the triangle inequality. Because all the coefficients of $\alpha_{2k-3},\ldots,\alpha_3$ were positive, the upper bound can be obtained by replacing α_n by α^0_n. From this and from the fact that I_0 was the equality function in (20) and therefore in (24) there follows

$$(28) \qquad |D_{2k-1}| \leqq -D^0_{2k-1}, \qquad k = 1,2,\ldots.$$

Because, except for $k = 2$, equality in (28) can be reached only if $|\alpha_3| = \alpha^0_3$ the only extremal function of this estimation for $k \neq 2$ is $I_0(z)$.

Thus we have proved

THEOREM 2. *For every function*

$$z = I(w) = w + \frac{D_1}{w} + \frac{D_3}{w^3} + \cdots,$$

inverse to odd Σ_b-functions, the estimations

$$(29) \qquad \begin{aligned} &|D_1| \leqq 1 - b^2 \\ &|D_{2n+1}| \leqq -\sum_{k=0}^{n} \beta_{nk} b^{2k} \end{aligned}$$

hold. Here

$$\beta_{nk} = (-1)^k \frac{2}{k!} \frac{2n-k}{2(n-k+1)} \frac{(2n-k-1)!}{[(n-k)!]^2}, \qquad n = 1,2,\ldots, \quad k = 1,2,\ldots,n.$$

The only extremal functions in (29) for all the indexes except $n = 1$ is the function satisfying the equation

$$I_0(w) + I_0(w)^{-1} = b^2 w^{-1} + w$$

and its rotations. For $n = 1$ the problem of the uniqueness of the extremal I_0 remains open.

The above result generalizes that obtained by Netanyahu [4] and Schober ([5], corollary, p. 116).

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